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Abelian Toda field theories on the noncommutative plane

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Abstract: Generalizations of $GL(n)$ abelian Toda and $\widetilde{GL}(n)$ abelian affine Toda field theories to the noncommutative plane are constructed. Our proposal relies on the noncommutative extension of a zero-curvature condition satisfied by algebra-valued gauge potentials dependent on the fields. This condition can be expressed as noncommutative Leznov-Saveliev equations which make possible to define the noncommutative generalizations as systems of second order differential equations, with an infinite chain of conserved currents. The actions corresponding to these field theories are also provided. The special cases of $GL(2)$ Liouville and $\widetilde{GL}(2)$ sinh/sine-Gordon are explicitly studied. It is also shown that from the noncommutative (anti-)self-dual Yang-Mills equations in four dimensions it is possible to obtain by dimensional reduction the equations of motion of the two-dimensional models constructed. This fact supports the validity of the noncommutative version of the Ward conjecture. The relation of our proposal to previous versions of some specific Toda field theories reported in the literature is presented as well.

Contents

1. Introduction

The research on Noncommutative Field theories (NCFT) has been very active since the appearance of these theories as low-energy limits of string theories in the presence of magnetic fields [1]. In the context of NCFT, noncommutative (nc) extensions of two-dimensional Integrable Field Theories have been investigated [2, 3, 4, 5, 6, 7, 8, 9, 10]. Since in two-dimensions a nc deformation of a model requires a noncommutative time-coordinate, the causality and unitarity properties of the theory can be compromised [11, 12]. However, it is conceivable that in exactly solvable systems this situation should be improved or even disappear, as discussed in [13]. In order to avoid the acasual behavior in the two-dimensional case, Euclidean models can be considered.

It is well known that the nc deformation of a theory is not unique since it is always possible to construct different nc extensions that will lead to the same commutative limit (see the nc generalizations of sine-Gordon model in [6]). In this sense, preserving the integrability properties of the theory can be a guiding principle in order to construct nc deformations of two-dimensional theories.

Following the previous direction, in [6] a nc extension of the zero-curvature condition was introduced. The nc extensions of integrable theories, constructed from this condition, have an infinite number of conserved charges which, however, not always guaranteed the complete classical integrability of the theory (see [7]). The amplitude for particle production processes vanishes exactly in an integrable model and that means that it vanishes to each order in a loop expansion, that is, in powers of $\hbar$. In particular it should vanish at tree-level, which corresponds to the classical limit of the theory, and is the hallmark of classical integrability. The existence of non-trivial conserved charges has another important consequence: multi-particle amplitudes are factorized into products of two-body processes [14]. In [7] was proved that the nc extension of the sine-Gordon model constructed in [5, 6] from a nc zero-curvature condition suffers from acasual behavior and it has a non-factorized S-matrix since particle production occurs. Therefore, we can be sure that this model is not classically integrable.

In the ordinary commutative case, it is well known that the integrable Conformal Toda (and Conformal Affine Toda) field theories can be obtained from the Wess-Zumino-Novikov-Witten ($WZNW$) [15] (two-loop $WZNW$) model via Hamiltonian Reduction [16, 17]. The algebraic structure of Toda theories is connected with a $G_0 \in G$ embedding of the G-invariant $WZNW$ (two-loop $WZNW$) model. Specifically abelian Toda theories are connected with abelian G_0 subgroups of G. In order to eliminate the degrees of freedom in the tangent space G/G_0 it is possible to implement constraints upon specific components of the chiral currents $J, \bar{J}$ of the $WZNW$ (two-loop $WZNW$) model. The equations of motion of the resultant model will be then defined in the G_0 subgroup. Usually these equations can be represented as a zero-curvature condition using the Leznov-Saveliev formulation [18]. The resultant Toda theory preserves also the original conformal symmetry of the $WZNW$ model. By the other side, the affine Toda models associated to loop algebras are not conformally invariant but it has been shown to be completely integrable [19] and also derivable from the Leznov-Saveliev equations [18]. These models are in fact a "gauge fixed" version of the Conformal Affine Toda models [23].

In this paper we use a nc zero-curvature condition expressed as a nc extension of Leznov-Saveliev equations [6] to construct nc integrable extensions of $SL(n)$ abelian Toda field theories and $\widetilde{SL}(n)$ abelian affine Toda field theories. In order to define the zero grade subgroup G_0 we have taken into account that the previous groups are not closed under the noncommutative product, so they should be extended to $GL(n)$ and $\widetilde{GL}(n)$ respectively. If we want to preserve the proper *algebra-group* relation this

extension must be also done at the level of the algebra. This consideration imply the introduction of an additional scalar field associated to the identity generator and which will not decouple in the equations of motion. We explicitly studied the $GL(2)$ Liouville and $\widetilde{GL}(2)$ sinh/sine-Gordon extensions. As we will see, any element of the zero grade subgroup can be parameterized in alternative ways, what leads to equivalent nc generalizations of the corresponding original models. This is a consequence of the nonabelian character of the zero grade subgroup in the nc setup. The models obtained as nc analogs of sine-Gordon reproduce already presented suggestions in [8] as nc extensions of this model. These generalizations [8] seem to retain some of the nice properties of the original sine-Gordon theory. They have an infinite chain of conserved charges, apparently a casual S-matrix, and particle production may be does not occur as was checked for some dispersion processes at tree level [8].

Another argument that supports the possible integrability of the nc Toda field theories constructed in this work is that all of them can be derived from the 4-dimensional nc self-dual Yang-Mills (SDYM) equations through a suitable dimensional reduction via the nc self-dual Chern-Simons (SDCS) system as we will see. The nc SDYM theory was shown in [45] to be classically integrable and in [24] was studied at the quantum level, where it was found that it has a factorized S-matrix. In this work we will also particularly see how the equations of motion that define the nc extensions of Liouville and sinh/sine-Gordon models can be obtained through a dimensional reduction process from 4-dimensional nc self-dual Yang-Mills in the Yang formulation. In this way all the models are derivable from an integrable field theory.

On the other hand, the nc extensions here constructed do not preserve the conformal invariance of the original abelian Conformal Toda theories. As it is well known, the introduction of a constant noncommutative parameter spoils the conformal symmetry. This symmetry has not been very well studied in the nc context, but it seems that in order to define a nc extension of this symmetry the deformation parameter should not be constant [22].

In the literature it has been reported different nc extensions of some specific Toda field theories: 1-sinh/sine-Gordon [5, 7, 6, 8, 9, 10], 2-Liouville [9, 10], 3-open Toda field chain or abelian Conformal Toda field theories [2, 9], 4- abelian affine Toda field theories or closed Toda field chain [9]. Up to now many of these extensions for the same models seem to be disconnected. With our formulation we are given a general framework where all these models are included as particular cases, excluding the proposals in [10]. Therefore we have extended these previous results, putting most of them on a more systematic, general and unifying footing.

The first nc extension of a Toda field theory was presented in [2]. It is well known that the abelian Conformal Toda theories when the fields only depend on the time coordinate can be modelled as a one-dimensional open chain of n particles with nonlinear nearest neighbor interaction [25]. The abelian affine corresponds

to closed chains. Deforming the bicomplex representation of the open Toda field chain it was constructed in [2] a nc extension of this theory up to first order in the noncommutative parameter θ. Here we will see how our nc extension of abelian Toda models reduces to the proposal in [2] when a perturvative expansion on the noncommutative parameter is considered up to first order. By the other side in [9] it was shown that using some simplifying algebraic ansatz the two-dimensional nc self-dual equations for the Chern-Simons solitons can be reduced to a nc extension of the Toda and affine Toda equations. The nc generalization for the Toda field theories proposed in [9] is presented as a system of first order differential equations that apparently could not be reduced to second order differential equations. In this work we will see how proposing a different ansatz it is possible to construct from the nc Chern-Simons self-dual system extended Toda field theories as second order differential equation systems and even to establish the compatibility with the suggestion in [9]. In this sense with our proposal we have nc versions of abelian and abelian affine Toda theories more treatable as physical theories since we also provide the corresponding actions. Moreover it is still possible to use the relation of the nc self-dual Chern-Simons equations to the nc chiral model for constructing the solutions of the models. Specifically this last point will be presented in [47]. There are other nc versions for the specific cases of Liuoville and sine-Gordon [10]. In this case the construction is based on a zero-curvature condition in terms of continual algebras [33]. A brief discussion about these models is also presented in this paper although they can not be obtained using our formalism. Let us mention that in [37] was also extended to the nc scenario the Toda hierarchy which is a generalization of the Toda lattice equations [38].

This paper is organized as follows. In the first section the nc Leznov-Saveliev equations are obtained by imposing appropriate constraints on the chiral currents of the $WZNW_\star$ model. In this section we also provide a nc action whose Euler-Lagrange equations of motion are the nc Leznov-Saveliev equations. Section 2 is devoted to the construction of the nc analogs of $GL(n)$ abelian Toda theories. Specially the relation to the nc open Toda field theory presented in [2] is discussed. The particular case of $GL(2)$ Liouville model is studied in the third part of this section. The fourth part contains the derivation of the equations of motion of nc Liouville from nc self dual Yang-Mills in the Yang formulation [45]. In the last part of section 3 the relation to previous proposals is discussed. Starting from the nc extension of the Leznov-Saveliev equations we construct in section 4 the nc generalization of $\widetilde{GL}(n)$ abelian affine Toda theories. In this section is also studied the special case of $\widetilde{GL}(2)$ sinh/sine-Gordon model. The derivation of its corresponding equations of motion from nc self dual Yang-Mills is presented as well. At the end of section 4 the relation to previous versions is included. In the last section it is shown how the nc Leznov-Saveliev equations [6] can be derived from the nc (anti-)self-dual Yang-Mills theory by a dimensional reduction process that has the nc self-dual Chern-Simons system as

an intermediate step. In this sense the connection of our nc extensions of abelian and abelian affine Toda theories to the proposal in [9] is established. Section 6 provides the conclusions and finally the appendix is dedicated to present some useful algebraic properties.

2. Constrained $WZNW_\star$ model

It is well known that Toda field theories connected with finite simple Lie algebras, on the ordinary commutative case, can be regarded as constrained Wess-Zumino-Novikov-Witten ($WZNW$) models [16]. By placing certain constraints on the chiral currents, the G-invariant $WZNW$ model reduces to the appropriate Toda field theory. Specifically, the abelian Toda field theories are connected with abelian embeddings $G_0 \subset G$. In this section we will see how this procedure also works on the nc setting.

Before we start, let us remind that usually a NCFT [28] is constructed from a given field theory by replacing the product of fields by an associative $\star$-product. Considering that the noncommutative parameter $\theta^{\mu\nu}$ is a constant antisymmetric tensor, the deformed product of functions is expressed trough the Moyal product [35]

$$\phi_1(x)\phi_2(x) \to \phi_1(x) \star \phi_2(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{x_1}\partial_\nu^{x_2}}\phi_1(x_1)\phi_2(x_2)|_{x_1=x_2=x}. \tag{2.1}$$

In the following we will refer to functions of operators in the noncommutative deformation by a $\star$ sub-index, for example $e_\star^\phi = \sum_{n=1}^{\infty}\frac{1}{n!}\phi_\star^n$ (the n-times star-product of ϕ is understood).

Consider now the nc generalization of the $WZNW$ model introduced in [27]

$$\begin{aligned} S_{WZNW_\star} = & -\frac{k}{4\pi}\int_\Sigma d^2z Tr(g^{-1}\star\partial g\star g^{-1}\star\bar{\partial}g) + \\ & \frac{k}{24\pi}\int_{\mathcal{B}} d^3x\epsilon_{ijk}(g^{-1}\star\partial_i g\star g^{-1}\star\partial_j g\star g^{-1}\star\partial_k g). \end{aligned} \tag{2.2}$$

Here $\mathcal{B}$ is a three-dimensional manifold whose boundary $\partial\mathcal{B} = \Sigma$. We are using the coordinates $z = t+x$, $\bar{z} = t-x$ and $\partial = \frac{1}{2}(\frac{\partial}{\partial t}+\frac{\partial}{\partial x})$, $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial t}-\frac{\partial}{\partial x})$ in the boundary, where $z, \bar{z}$ or equivalently x, t are noncommutative, but the extended coordinate y on the manifold $\mathcal{B}$ remains commutative, i.e. $[z,\bar{z}] = \theta$, $[y,z] = [y,\bar{z}] = 0$. The Euler-Lagrange equations of motion corresponding to (2.2) are

$$\bar{\partial}J = \partial\bar{J} = 0, \tag{2.3}$$

where J and $\bar{J}$ represent the conserved chiral currents

$$J = g^{-1}\star\partial g, \qquad \bar{J} = -\bar{\partial}g\star g^{-1}. \tag{2.4}$$

The fields α_a parameterize the group element $g \in G$ through $g = e_\star^{\alpha_a T_a}$, where T_a are the generators of the corresponding Lie algebra $\mathcal{G}$. It is our interest to define the

theories in a G_0 subgroup of G. So, we would like to eliminate the unwanted degrees of freedom that correspond to the tangent space G/G_0. In order to that be achieved we will implement constraints upon specific components of the currents $J, \bar{J}$. In this way we will show that the usual procedure [16] works equally well on the nc setting.

Suppose we have defined a grading operator Q in the algebra $\mathcal{G}$ that decomposes it in $\mathbb{Z}$-graded subspaces, say

$$[Q, \mathcal{G}_i] = i\mathcal{G}_i, \qquad [\mathcal{G}_i, \mathcal{G}_j] \in \mathcal{G}_{i+j}. \tag{2.5}$$

This means that the algebra $\mathcal{G}$ can be represented as the direct sum,

$$\mathcal{G} = \bigoplus_i \mathcal{G}_i. \tag{2.6}$$

The subspaces $\mathcal{G}_0, \mathcal{G}_>, \mathcal{G}_<$ are subalgebras of $\mathcal{G}$, composed of the Cartan and of the positive/negative steps generators respectively. The algebra can then be written using the triangular decomposition,

$$\mathcal{G} = \mathcal{G}_< \bigoplus \mathcal{G}_0 \bigoplus \mathcal{G}_>. \tag{2.7}$$

Denote the subgroup elements obtained through the $\star$-exponentiation of the generators of the corresponding subalgebras as

$$N = e_\star^{\mathcal{G}_<}, \quad B = e_\star^{\mathcal{G}_0}, \quad M = e_\star^{\mathcal{G}_>}. \tag{2.8}$$

Proposing a nc Gauss-like decomposition, an element g of the nc group G can be expressed as

$$g = N \star B \star M. \tag{2.9}$$

Introducing (2.9) in (2.4), the chiral currents $J, \bar{J}$ read,

$$J = M^{-1} \star K \star M, \qquad \bar{J} = -N \star \bar{K} \star N^{-1}, \tag{2.10}$$

where

$$\begin{aligned} K &= B^{-1} \star N^{-1} \star \partial N \star B + B^{-1} \star \partial B + \partial M \star M^{-1}, \\ \bar{K} &= N^{-1} \star \bar{\partial} N + \bar{\partial} B \star B^{-1} + B \star \bar{\partial} M \star M^{-1} \star B^{-1}. \end{aligned} \tag{2.11}$$

With the chiral currents (2.10), the equations of motion (2.3) transform to

$$\begin{aligned} \bar{\partial} K + [K, \bar{\partial} M \star M^{-1}]_\star &= 0, \\ \partial \bar{K} - [\bar{K}, N^{-1} \star \bar{\partial} N]_\star &= 0. \end{aligned} \tag{2.12}$$

The reduced model is defined by giving the constant elements $\epsilon_\pm$ of grade ± 1, which are responsible for constraining the currents in a general manner to [1]

$$J_{constr} = j + \epsilon_-, \qquad \bar{J}_{constr} = \bar{j} + \epsilon_+, \tag{2.13}$$

[1]See [16] for the commutative case.

where $j, \bar{j}$ contain generators of grade zero and positive, and zero and negative respectively. The effect of the constraints on the chiral currents $J, \bar{J}$ (2.10) translates in the conditions,

$$B^{-1} \star N^{-1} \star \partial N \star B|_{constr} = \epsilon_-,$$
$$B \star \bar{\partial} M \star M^{-1} \star B^{-1}|_{constr} = \epsilon_+, \tag{2.14}$$

because from the graded structure these are the only terms in (2.11) that contain generators of negative and positive grade respectively. As result of the reduction process, the degrees of freedom in M, N are eliminated and the equations of motion of the constrained model are natural nc extensions of the Leznov-Saveliev equations of motion [18], namely

$$\bar{\partial}(B^{-1} \star \partial B) + [\epsilon_-, B^{-1} \star \epsilon_+ B]_\star = 0,$$
$$\partial(\bar{\partial} B \star B^{-1}) - [\epsilon_+, B\epsilon_- \star B^{-1}]_\star = 0. \tag{2.15}$$

Both equations are equivalent. One can see that by $\star$-multiplying B from the left and B^{-1} from the right the first equation in (2.15), say

$$B \star \{\bar{\partial}(B^{-1} \star \partial B) + [\epsilon_-, B^{-1} \star \epsilon_+ B]_\star = 0\} \star B^{-1},$$
$$-\bar{\partial} B \star B^{-1} \star \partial B \star B^{-1} + \bar{\partial}\partial B \star B^{-1} - [\epsilon_+, B\epsilon_- \star B^{-1}]_\star = 0. \tag{2.16}$$

Then, using that $\partial(B \star B^{-1}) = 0$ the second equation in (2.15) is obtained, what means that both equations are simultaneously satisfied. In [6] these equations were used to define a nc extension of the sinh/sine-Gordon model. In contrast to the previous suggestion [6], in this paper we propose an alternative definition for B which preserves the proper *algebra-group* relation. This choice will lead to a nc sine-Gordon defined as a system of two coupled second order equations for two scalar fields that reduced to sine-Gordon model and a free scalar field in the commutative limit. The nc extensions of the Leznov-Saveliev equations (2.15) for $GL(2)$ were also obtained in [42] from the nc generalization of the $SL(2)$ affine Toda model coupled to matter (Dirac) fields.

As shown in [6], the equations of motion (2.15) can be expressed as a generalized $\star$-zero-curvature condition

$$\bar{\partial} A - \partial \bar{A} + [A, \bar{A}]_\star = 0, \tag{2.17}$$

since the potentials are taken as

$$A = -B\epsilon_- \star B^{-1}, \qquad \bar{A} = \epsilon_+ + \bar{\partial} B \star B^{-1}. \tag{2.18}$$

The $\star$-zero-curvature condition (2.17) implies the existence of an infinite amount of conserved charges [6]. For this reason in order to preserve the original integrability properties of the two-dimensional Toda models (2.15) can be a reasonable starting point for constructing the nc analogs.

2.1 The noncommutative action

It is not difficult to propose a nc action from which the nc Leznov-Saveliev equations (2.15) can be derived. In fact, (2.15) are the Euler-Lagrange equations of motion of the action

$$S = S_{WZNW_\star}(B) + \frac{k}{2\pi}\int d^2 z Tr(\epsilon_+ B\epsilon_- \star B^{-1}). \tag{2.19}$$

This is the nc generalization of the effective action obtained from the $WZNW$ model gauging the degrees of freedom in M, N and integrating over the corresponding gauge fields [17]. In [36] different gauged $WZNW_\star$ models were constructed. However, the integration over the gauge fields on the nc scenario requires special care. For this reason in the present paper we limit to propose the action (2.19) as corresponding to the equations (2.15) and it remains to be proved if (2.19) can be obtained from (2.2) by a gauging procedure.

3. NC $GL(n)$ abelian Toda field theories

In the following we will construct nc analogs of the $SL(n)$ abelian Toda field theories, the simplest class of Toda models which are completely integrable and conformal invariant. As we already mentioned, these models correspond to an abelian subgroup $G_0 \subset G$. Considering the extension of the $\mathcal{SL}(n)$ algebra to $\mathcal{GL}(n)$, what is necessary in order to obtain a nc closed group, the gradation operator

$$Q = \sum_{i=1}^{n-1} \frac{2\lambda_i \cdot H}{\alpha_i^2}, \tag{3.1}$$

defines the subalgebra of grade zero $\mathcal{G}_0 = U(1)^n = \{I, h_i, \ \ i = 1 \dots n-1\}$, where the Cartan generators are defined in the Chevalley basis as $h_i = \frac{2\alpha_i \cdot H}{\alpha_i^2}$. In (3.1) H represents the Cartan subalgebra, α_i is the i^{th} simple root and λ_i is the i^{th} fundamental weight that satisfies $\frac{2\lambda_i \cdot \alpha_j}{\alpha_i^2} = \delta_{ij}$. The zero grade group element B is then expressed through the $\star$-exponentiation of the generators of the zero grade subalgebra $\mathcal{G}_0$, i.e. the $\mathcal{SL}(n)$ Cartan subalgebra plus the identity generator,

$$B = e_\star^{\Sigma_{i=1}^{n-1}\varphi_i h_i + \varphi_0 I}. \tag{3.2}$$

Notice that the zero grade subgroup G_0 despite it is spanned by the generators of the Cartan subalgebra, turns out to be nonabelian, i.e, if g_1, g_2 are two elements of the zero grade subgroup G_0 then $g_1 \star g_2 \neq g_2 \star g_1$. For this reason *abelian* makes reference to the property of the original theory.

The constant generators of grade ± 1 are chosen as

$$\epsilon_\pm = \Sigma_{i=1}^{n-1} \mu_i E_{\pm\alpha_i}, \tag{3.3}$$

where $E_{\pm\alpha_i}$ are the steps generators associated to the positive/negative simple roots of the algebra and μ_i are constant parameters.

Let us consider the $n \times n$ matrix representation

$$(h_i)_{\mu\nu} = \delta_{\mu\nu}(\delta_{i,\mu} - \delta_{i+1,\mu}),\ (E_{\alpha_i})_{\mu\nu} = \delta_{\mu,i}\delta_{\nu,i+1},\ (E_{-\alpha_i})_{\mu\nu} = \delta_{\nu,i}\delta_{\mu,i+1}. \tag{3.4}$$

It is not difficult to see that in this case the zero grade group element (3.2) can be represented by the $n \times n$ diagonal matrix,

$$B = \begin{pmatrix} e_\star^{\varphi_1+\varphi_0} & 0 & 0 & 0 & \dots & 0 \\ 0 & e_\star^{-\varphi_1+\varphi_2+\varphi_0} & 0 & 0 & \dots & 0 \\ 0 & 0 & e_\star^{-\varphi_2+\varphi_3+\varphi_0} & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & e_\star^{\varphi_{n-1}-\varphi_{n-2}+\varphi_0} & 0 \\ 0 & 0 & 0 & \dots & 0 & e_\star^{-\varphi_{n-1}+\varphi_0} \end{pmatrix}. \tag{3.5}$$

Let us now introduce the variables,

$$\begin{aligned} \varphi_1 + \varphi_0 &= \phi_1, \\ -\varphi_k + \varphi_{k+1} + \varphi_0 &= \phi_{k+1}, \quad \text{for } k = 1 \text{ to } n-2, \\ -\varphi_{n-1} + \varphi_0 &= \phi_n. \end{aligned} \tag{3.6}$$

In these new fields the components of the gauge connections (2.18) are written as

$$\bar{A}_{ij} = \bar{\partial}(e_\star^{\phi_i}) \star e_\star^{-\phi_i}\delta_{ij} + \mu_i\delta_{i+1,j} \quad \text{and} \quad A_{ij} = -\mu_i e_\star^{\phi_{i+1}} \star e_\star^{-\phi_i}\delta_{i,j+1}, \tag{3.7}$$

with $i, j = 1 \dots n$. One can now introduce the gauge potentials (3.7) in the $\star$-zero-curvature condition (2.17) to obtain the equations of motion that define the nc extension of abelian Toda models,

$$\partial(\bar{\partial}(e_\star^{\phi_k}) \star e_\star^{-\phi_k}) = \mu_k^2 e_\star^{\phi_{k+1}} \star e_\star^{-\phi_k} - \mu_{k-1}^2 e_\star^{\phi_k} \star e_\star^{-\phi_{k-1}}, \tag{3.8}$$

a system of n-coupled equations ($k = 1 \dots n$). Notice that the diagonal elements of the matrix equation (2.17) are the equations of motion and the off-diagonal elements vanish. In (3.8) for the first and last equation $\mu_0 = \mu_n = 0$ and $\phi_0 = \phi_{n+1} = 0$. In order to compute the commutative limit we introduce the original fields $\varphi_0, \varphi_1, \dots \varphi_{n-1}$ (3.6) and then we apply the limit $\theta \to 0$, which transforms $e_\star^{\phi} \to e^{\phi}$. At the end we find, as expected, that the field φ_0 decouples, i.e. (3.8) leads to

$$\begin{aligned} \partial\bar{\partial}\varphi_i &= \mu_i^2 e^{-k_{ij}\varphi_j}, \qquad \text{for} \quad i = 1 \dots n-1, \\ n\partial\bar{\partial}\varphi_0 &= 0, \end{aligned} \tag{3.9}$$

where the first $n-1$ equations become the usual abelian Conformal Toda equations with k_{ij} the Cartan matrix.

The action, whose Euler-Lagrange equations of motion leads to (3.8), can be obtained from (2.19) with (3.3) and (3.5). It reads

$$S(\phi_1,\ldots,\phi_n) = \sum_{k=1}^{n} S_{WZNW_\star}(e_\star^{\phi_k}) + \frac{k}{2\pi}\int d^2z \sum_{k=1}^{n-1}\mu_k^2(e_\star^{\phi_{k+1}}\star e_\star^{-\phi_k}), \qquad (3.10)$$

that in the commutative limit yields

$$S(\varphi_1,\ldots,\varphi_{n-1},\varphi_0) = S_{CT}(\varphi_1,\ldots,\varphi_{n-1}) + nS_0(\varphi_0) \qquad (3.11)$$

where

$$\begin{aligned} S_{CT}(\varphi_1,\ldots,\varphi_{n-1}) &= -\frac{k}{4\pi}\int d^2z(k_{ij}\partial\varphi_i\bar{\partial}\varphi_j - 2\sum_{i=1}^{n-1}\mu_i^2 e^{-k_{ij}\varphi_j}), \\ S_0(\varphi_0) &= -\frac{k}{4\pi}\int d^2z\partial\varphi_0\bar{\partial}\varphi_0. \end{aligned} \qquad (3.12)$$

This is the action of the abelian Conformal Toda models plus the corresponding kinetic term for the free field φ_0. In the last calculation we have made use of the nc generalization of the Polyakov-Wiegmann identity

$$\begin{aligned} S_{WZNW_\star}(g_1\star g_2) = {} & S_{WZNW_\star}(g_1) + S_{WZNW_\star}(g_2) - \\ & -\frac{1}{2\pi}\int dz d\bar{z} Tr(g_1^{-1}\star\partial g_1\star\bar{\partial}g_2\star g_2^{-1}). \end{aligned} \qquad (3.13)$$

The action (3.10) has the left-right local symmetry

$$e_\star^{\phi_k} \to h_0(z)\star e_\star^{\phi_k(z,\bar{z})}\star\tilde{h}_0(\bar{z}), \quad \text{for} \quad k=1\ldots n, \qquad (3.14)$$

where $h_0(z),\tilde{h}_0(\bar{z}) \in G_0$, which is relic of the left-right local symmetry $g(z,\bar{z}) \to h(z)\star g(z,\bar{z})\star\tilde{h}(\bar{z})$ with $h(z),\tilde{h}(\bar{z}) \in G$ of the $WZNW_\star$ model, whose corresponding conserved currents (2.4) close, in the same way as the ordinary commutative case, a Kac-Moody algebra [36]. The currents of the abelian Conformal Toda models (3.12) generate a W-algebra [34]. Since we have now in our action an infinite number of time derivatives it is not trivial how to define the conjugate momenta associated to the fields ϕ_k and consequently to define the corresponding Poisson brackets necessary to study the algebra of currents of the constrained model. Consider the special symmetry subgroup $U_L(1)\times U_R(1)$ where $h(z) = U_L(z) = e_\star^{i\alpha_1(z)}$ and $\tilde{h}(\bar{z}) = U_R(\bar{z}) = e_\star^{i\alpha_2(\bar{z})}$ and $\phi_k \to i\phi_k$. As far as the global symmetry is concerned there is no difference between the point-wise and star product, so only the combination $U_LU_R = e^{i\alpha}$ is important. Notice that this is a symmetry of the action as well as of the equations of motion. Let us try to find the corresponding conserved charge and for this purpose we will localize α. Since the action is invariant for this variation of the field

$$\delta S \equiv -\int d^2x J^\mu(\phi)\partial_\mu\alpha(x) = 0. \qquad (3.15)$$

Integrating by parts

$$\int d^2x \partial_\mu J^\mu \alpha(x) = \int d^2x [f,g]_\star = 0, \tag{3.16}$$

for some functions f, g. What means that here the notions of conserved current and of conserved charged are not well-defined. So it seems that Noether's theorem when time is a not-commuting coordinate no longer applies. For this reason it is not clear what are the conserved charges associated to these symmetries if there is any one. The study of the symmetries of these theories requires first a nc analog of Noether theorem that could relate in a possible way general symmetries to conservation laws.

So, in this section we have constructed nc analogs of abelian Toda field theories from a $\star-$zero-curvature condition and in this way they posses an infinite number of conserved charges. Associating to these theories an appropriate bicomplex it is possible to construct by an iterative process an infinite chain of conserved charges [2]. In the next section, in relation to the work [2], we will study the corresponding bicomplex. Nevertheless the study of the influence of this infinite chain of conserved charges on the integrability properties of the theories is still an open question.

3.1 The nc Abelian Toda field theories as open Toda field chains

The open Toda chains on the usual commutative case are integrable systems associated with finite-dimensional Lie algebras. The model consists of a one-dimensional chain of n particles with nonlinear nearest neighbor interaction [25]. The relativistic invariant abelian Conformal Toda field theories (3.9) can be called as open Toda field chains since the equations of motion of these theories reduce to the open Toda chain equations of motion when the fields do not depend on x. To the Toda field theories it is possible in general to associate a bicomplex [39], which is a special case of zero curvature formulation. Noncommutative extensions of integrable models can be obtained deforming the associated bicomplex by replacing the ordinary products of functions with the Moyal product [2]. Following this procedure in [2] was constructed a nc extension of a Toda field theory on an open finite one-dimensional lattice up to first order in θ. It is not difficult to see that expressing the components of the zero grade element as $e_\star^{\phi_k} = e^{q_k}(1 + \theta \tilde{q}_k) + O(\theta^2)$ our system (3.8) reduces to the proposal of Dimakis- Müller-Hoissen (expressions (3.10) and (3.11) of [2] mapping $q_k, \tilde{q}_k \to -q_k, -\tilde{q}_k$ respectively). At zero order of θ the open Toda equations,

$$\begin{aligned} (\partial_t^2 - \partial_x^2) q_k = e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}} \quad \text{for} \quad k = 2...n-1, \\ (\partial_t^2 - \partial_x^2) q_1 = e^{q_2 - q_1}, \qquad (\partial_t^2 - \partial_x^2) q_n = -e^{q_n - q_{n-1}}, \end{aligned} \tag{3.17}$$

are obtained. Then at first order of θ the corresponding equations are

$$\begin{aligned} (\partial_t^2 - \partial_x^2)\tilde{q}_1 &= \{\partial_t q_1, \partial_x q_1\} + e^{q_2 - q_1}(\tilde{q}_2 - \tilde{q}_1), \\ (\partial_t^2 - \partial_x^2)\tilde{q}_k &= \{\partial_t q_k, \partial_x q_k\} + e^{q_{k+1}-q_k}(\tilde{q}_{k+1} - \tilde{q}_k) - e^{q_k - q_{k-1}}(\tilde{q}_k - \tilde{q}_{k-1}), \\ \partial_t^2 - \partial_x^2)\tilde{q}_n &= \{\partial_t q_n, \partial_x q_n\} + e^{q_n - q_{n-1}}(\tilde{q}_n - \tilde{q}_{n-1}), \end{aligned} \tag{3.18}$$

where we have considered $\mu_k = \frac{1}{2}$ for $k = 1...n-1$ and $\{f,g\} = \partial_t f \partial_x g - \partial_x f \partial_t g$. This fact is not unexpected since the bicomplex equation from where the Toda chain equations are derived can be written as the nc Leznov-Saveliev equation (2.15) as we will see immediately. Consider the bicomplex equation [2]

$$M_t - M_x = L \star S - S \star L, \tag{3.19}$$

with

$$L = G^{-1} \star S^T \star G, \quad M = 2G^{-1} \star \partial G, \tag{3.20}$$

G an invertible $n \times n$ matrix that in [2] is taken diagonal and S^T the transpose of

$$S = \sum_{i=1}^{n-1} E_{i,i+1} \quad \text{with} \quad (E_{i,j})^k l = \delta_i^k \delta_{j,l}. \tag{3.21}$$

Taking into account that the matrix S in this case is nothing more than the matrix representation of the constant generators of grade ± 1 (3.3) and G the matrix representation of the zero grade group element B, the equation (3.19) can be written as a nc Leznov-Saveliev equation (2.15) [2]. In this sense we have the bicomplex associated to these integrable models what allows to construct generalized conserved densities. In this specific case the first two charges were computed in [2]. The generalization to the closed Toda chain related to loop algebras and consequently to the abelian affine Toda models is straightforward.

3.2 NC Liouville

Let us concentrate in this part of the section on an specific example of a Toda field theory, i.e. the Liouville model. This is an integral and conformal invariant theory that appears in many applications related to string theory and two-dimensional quantum gravity. It turns out to be the simplest example of an abelian Conformal Toda model, connected to $SL(2)$. Following that we can present a nc extension of this model taking $n = 2$ in (3.8), say

$$\begin{aligned} \partial(\bar{\partial}(e_\star^{\phi_+}) \star e_\star^{-\phi_+}) &= \mu^2 e_\star^{\phi_-} \star e_\star^{-\phi_+}, \\ \partial(\bar{\partial}(e_\star^{\phi_-}) \star e_\star^{-\phi_-}) &= -\mu^2 e_\star^{\phi_-} \star e_\star^{-\phi_+}, \end{aligned} \tag{3.22}$$

where we have called $\phi_+ = \phi_1$, $\phi_- = \phi_2$ and $\mu_1 = \mu$. One can now also compute the sum and difference of that equations to find

$$\begin{aligned} \partial(\bar{\partial}(e_\star^{\phi_+}) \star e^{-\phi_+} + \bar{\partial}(e_\star^{\phi_-}) \star e^{-\phi_-}) &= 0, \\ \partial(\bar{\partial}(e_\star^{\phi_+}) \star e^{-\phi_+} - \bar{\partial}(e_\star^{\phi_-}) \star e^{-\phi_-}) &= 2\mu^2 e_\star^{\phi_-} \star e_\star^{-\phi_+}. \end{aligned} \tag{3.23}$$

[2]In fact consider in this case $G = B^{-1}$ and $\epsilon_+ = \frac{1}{2}S$.

In this sense (3.22) or (3.23) will represent the nc analogs of the Liouville model. Both systems in the commutative limit will lead to a decoupled model of two fields

$$\partial\bar{\partial}\varphi_0 = 0 \quad \text{and} \quad \partial\bar{\partial}\varphi_1 = \mu^2 e^{-2\varphi_1}, \tag{3.24}$$

as we already saw in (3.9). The first equation in (3.23) transforms to a free field equation for φ_0 and the second leads to the usual Liouville equation.

The action corresponding to the nc analog of Liouville model follows from (3.10) for $n = 2$, namely

$$S(\phi_+, \phi_-) = S_{WZNW_\star}(e_\star^{\phi_+}) + S_{WZNW_\star}(e_\star^{\phi_-}) + \frac{k}{2\pi}\int d^2z\mu^2 e_\star^{\phi_-} \star e_\star^{-\phi_+}. \tag{3.25}$$

The nonabelian character of the zero grade subgroup allows an alternative parameterization for the zero grade group element B, say

$$B = e_\star^{\varphi_1 h} \star e_\star^{\varphi_0 I}, \tag{3.26}$$

where h is the Cartan and I is the identity generator. We can keep the same constant generators of grade ± 1,

$$\epsilon_\pm = \mu E_{\pm\alpha}. \tag{3.27}$$

Introduce (3.26) and (3.27) in the nc generalization of Leznov-Saveliev equations (2.15). Using algebraic properties it is very simple to see that

$$B \star \epsilon_- B^{-1} = e_\star^{-2\varphi_1}\epsilon_-. \tag{3.28}$$

For computing the equations of motion, notice that the matrix representation (3.4) reduces to the usual representation

$$E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.29}$$

for $\mathcal{SL}(2)$. In this way one find that the equations that define the nc extension of Liouville model in this alternative parameterization are

$$\begin{aligned} \partial(\bar{\partial}(e_\star^{\varphi_1} \star e_\star^{\varphi_0}) \star e_\star^{-\varphi_0} \star e_\star^{-\varphi_1}) &= \mu^2 e_\star^{-2\varphi_1}, \\ \partial(\bar{\partial}(e_\star^{-\varphi_1} \star e_\star^{\varphi_0}) \star e_\star^{-\varphi_0} \star e_\star^{\varphi_1}) &= -\mu^2 e_\star^{-2\varphi_1}. \end{aligned} \tag{3.30}$$

One can also combine the previous equations to find the system of coupled second order equations,

$$\begin{aligned} \partial(\bar{\partial}(e_\star^{\varphi_1} \star e_\star^{\varphi_0}) \star e_\star^{-\varphi_0} \star e_\star^{-\varphi_1} + \bar{\partial}(e_\star^{-\varphi_1} \star e_\star^{\varphi_0}) \star e_\star^{-\varphi_0} \star e_\star^{\varphi_1}) &= 0, \\ \partial(\bar{\partial}(e_\star^{\varphi_1} \star e_\star^{\varphi_0}) \star e_\star^{-\varphi_0} \star e_\star^{-\varphi_1} - \bar{\partial}(e_\star^{-\varphi_1} \star e_\star^{\varphi_0}) \star e_\star^{-\varphi_0} \star e_\star^{\varphi_1}) &= 2\mu^2 e_\star^{-2\varphi_1}, \end{aligned} \tag{3.31}$$

which are more suitable for applying the commutative limit. When $\theta \to 0$ that system easily reduced to (3.24).

One can write an action for the nc Liouville model (3.30) using the general expression (2.19) and making use of the nc generalization of the Polyakov-Wiegmann identity (3.13). For this purpose, introduce (3.26) and (3.27) in (2.19) to obtain the corresponding action of nc Liouville (3.30)

$$S(\varphi_1,\varphi_0) = 2S_{PC_\star}(e_\star^{\varphi_1}) + 2S_{WZNW_\star}(e_\star^{\varphi_0}) + \frac{k}{2\pi}\int d^2z\mu^2 e_\star^{-2\varphi_1} - \\ -\frac{k}{2\pi}\int d^2z(\bar{\partial}e_\star^{\varphi_0}\star e_\star^{-\varphi_0}\star(e_\star^{-\varphi_1}\star\partial e_\star^{\varphi_1} + e_\star^{\varphi_1}\star\partial e_\star^{-\varphi_1})), \qquad (3.32)$$

where using the notation of [8] we have defined

$$S_{PC_\star}(g) = -\frac{k}{4\pi}\int_\Sigma d^2zTr(g^{-1}\star\partial g\star g^{-1}\star\bar{\partial}g). \qquad (3.33)$$

Looking at the action (3.32) it is noticeable the presence of the topological term of $WZNW_\star$, in this case shifted to the φ_0 field. This situation contrasts with the ordinary commutative case, where for Liouville and more generally for any abelian subspace this term equals zero.

The parameterizations (3.26) and (3.24) for the element B belonging to the zero grade subgroup in the usual commutative case are identical, but in a nc space-time they lead to equivalent models. Taking

$$e_\star^{\varphi_1}\star e_\star^{\varphi_0}\to e_\star^{\phi_+} \quad \text{and} \quad e_\star^{-\varphi_1}\star e_\star^{\varphi_0}\to e_\star^{\phi_-}, \qquad (3.34)$$

(3.30) can be transformed in (3.22). It is interesting to note that the usual Liouville singular solution [16]

$$\varphi_1 = \ln\cos(\alpha z - \beta\bar{z}) \qquad (3.35)$$

is also solution of the systems (3.22) and (3.30) as far as $\varphi_0 = \alpha z - \beta\bar{z}$ and the constant parameters α,β satisfy $\mu^2 = \alpha\beta$. This is a consequence of the fact that for this particular dependence of the fields φ_1,φ_0 on the variables $z,\bar{z}$ the star-product reduce to the usual product. So in this case φ_0 decouples and the nc model is reduced to the usual Liouville theory plus the equation for a free field. Other solutions of this model will be discussed in [47].

3.3 NC Liouville from nc self-dual Yang-Mills

It has been known for a long time that many two-dimensional integrable models can be obtained from the Yang-Mills self-duality equations in four dimensions by reductions [43]. Recently a nc extension of the self-dual Yang-Mills (NCSDYM) equations in the Yang formulation [45],

$$\partial_y(\partial_{\bar{y}}J\star J^{-1}) + \partial_z(\partial_{\bar{z}}J\star J^{-1}) = 0, \qquad (3.36)$$

has been proposed. In (3.36) $y,\bar{y},z,\bar{z}$ are complex independent variables $y = x_1 + ix_2$, $\bar{y} = x_1 - ix_2$, $z = x_3 - ix_4$, $\bar{z} = x_3 + ix_4$ that do not commute. Moreover,

from (3.36) through a dimensional reduction process some nc extensions of two-dimensional integrable models have been obtained [4, 7, 8]. In this part of the section we will show how the systems of equations (3.22) and (3.30) that define the nc extensions of Liouville model can be also derived from (3.36) through a suitable dimensional reduction. For this purpose consider [3]

$$J = e_\star^{wE_\alpha} \star B \star e_\star^{-wE_{-\alpha}}, \tag{3.37}$$

where $w = \mu(y + \bar{y})$ with μ a constant parameter and B a zero grade group element in this case of $GL(2, \mathbb{C})$, whose fields depend only on the variables $z, \bar{z}$.

Let us denote $M = e_\star^{wE_\alpha}$. It is straightforward to see that (3.36) reduces to

$$M \star \{\partial_z(\partial_{\bar{z}} B \star B^{-1}) - \mu^2 (E_\alpha B E_{-\alpha} \star B^{-1} - B E_{-\alpha} \star B^{-1} E_\alpha)\} \star M^{-1} = 0. \tag{3.38}$$

Remembering that for Liouville $\epsilon_\pm = \mu E_{\pm\alpha}$, the previous equation will render

$$\partial_z(\partial_{\bar{z}} B \star B^{-1}) - [\epsilon_+, B\epsilon_- \star B^{-1}]_\star = 0. \tag{3.39}$$

Taking $x_3 = t$ and $x_4 = ix$ this is the nc Leznov-Saveliev equation (2.15) from where (3.30) and (3.22) were derived. In this sense, we have shown that the nc analogs of Liouville theory (3.22) and (3.30) can be obtained through an appropriate reduction of the nc self-dual Yang-Mills system in the same way as in the ordinary commutative case. More general, in this work we will see how the nc Leznov-Saveliev equations (2.15) can be obtained through a dimensional reduction process from the nc (anti-)self-dual Yang-Mill equations via the nc self-dual Chern-Simons system.

3.4 Other proposals

As was mentioned in the previous work [6] the nc extension of a field theory is not unique. In this sense a different nc proposal for the Liouville model was presented in [10]

$$\bar{\partial}\left((e_\star^{\beta\phi})_{\star L}^{-1} \star \partial e_\star^{\beta\phi}\right) = e_\star^{\beta\phi}, \tag{3.40}$$

where $(e_\star^{\beta\phi})_{\star L}^{-1}$ is denoted as the left inverse function of $e_\star^{\beta\phi}$ with respect to the $\star$-product and β a constant. This equation is obtained from a zero-curvature condition on the basis of a generalization of Saveliev-Vershik continual Lie algebras [33]. Note that in this case the model is defined through only one equation. Not presented in [10], but it can be obtained as the Euler-Lagrange equation of motion of the action

$$S_{L-1} = S_{WZNW_\star}(e_\star^{\beta\phi}) + \frac{k}{2\pi} \int d^2 z e_\star^{\beta\phi}, \tag{3.41}$$

which can be thought as the naive nc generalization of the $SL(2)$ Toda theory at the level of the action just generalizing the derivative term. In the case of sinh/sine

[3] This is a nc extension of the J decomposition proposed in [46] for Liouville.

Gordon model studied in [6] this procedure leads to a *non*-integrable deformation since the amplitude for the scattering $2 \to 4$ process it is non-zero. This fact is in some sense explained by the difficulty to find a zero curvature representation for this deformation without the inclusion of extra conditions. In the case of (3.41) we failed to find a zero-curvature representation in terms of the Lie algebra $SL(2)$. But this does not mean that the model is not integrable. In order to check the integrability properties of (3.41) could be interesting to study the properties of the corresponding S-matrix. By the other side a different nc extension

$$\begin{aligned} \partial(\bar{\partial}(e_\star^{i\varphi_1}) \star e_\star^{-i\varphi_1}) &= \mu^2 e_\star^{-2i\varphi_1}, \\ \partial(\bar{\partial}(e_\star^{-i\varphi_1}) \star e_\star^{i\varphi_1}) &= -\mu^2 e_\star^{-2i\varphi_1}, \end{aligned} \tag{3.42}$$

could in principle be constructed from the $\star$-zero-curvature condition (2.17) for the algebra $SL(2)$, excluding the direction of the identity generator of the Cartan subalgebra (in this specific case we have considered that $\varphi_1 \to i\varphi_1$). In order to find an action for (3.22) or equivalently for the system

$$\begin{aligned} \partial(\bar{\partial}(e_\star^{i\varphi_1}) \star e_\star^{-i\varphi_1}) + \partial(\bar{\partial}(e_\star^{-i\varphi_1}) \star e_\star^{i\varphi_1}) &= 0, \\ \partial(\bar{\partial}(e_\star^{i\varphi_1}) \star e_\star^{-i\varphi_1}) - \partial(\bar{\partial}(e_\star^{-i\varphi_1}) \star e_\star^{i\varphi_1}) &= 2\mu^2 e_\star^{-2i\varphi_1}, \end{aligned} \tag{3.43}$$

it can be followed the procedure in [7]. So one can consider that if φ_1 is a complex field then $e_\star^{i\varphi_1^\dagger}$ satisfies the equations

$$\begin{aligned} \partial(\bar{\partial}(e_\star^{i\varphi_1^\dagger}) \star e_\star^{-i\varphi_1^\dagger}) &= -\mu^2 e_\star^{2i\varphi_1^\dagger}, \\ \partial(\bar{\partial}(e_\star^{-i\varphi_1^\dagger}) \star e_\star^{i\varphi_1^\dagger}) &= \mu^2 e_\star^{2i\varphi_1^\dagger}, \end{aligned} \tag{3.44}$$

obtained by hermitian conjugation of (3.42). An action for this model could be

$$S_{L-2} = S_{WZNW_\star}(e_\star^{i\varphi_1}) + S_{SWZN_\star}(e_\star^{-i\varphi_1^\dagger}) + \frac{k}{2\pi}\int d^2z (e_\star^{i2\varphi_1^\dagger} + e_\star^{-i2\varphi_1}), \tag{3.45}$$

if as independent equations are considered the first in (3.42) and the second in (3.44). In [9] was proposed another nc extension of the Liouville model starting from the nc self-dual Chern-Simons equations. We will discuss about this point later in this paper.

4. NC $\widetilde{GL}(n)$ abelian affine Toda field theories

Among Toda theories the affine models are quite special due, essentially, to the presence of soliton type solutions. Looking at possible applications in particle physics and in condensed matter, in this section we will extend these theories to the nc scenario.

Consider now the nc generalization of the two-loop $WZNW$ model [4], which is formally described by the same action (2.2), but in this case $\mathcal{G}$ is an infinite-dimensional algebra (see the appendix). In the same way N and M are infinite-dimensional subalgebras. The zero grade subgroup G_0 is however chosen to be finite-dimensional. Imposing appropriate constraints on the chiral currents (2.13) in such a way that now the constant generators of grade ± 1 include extra affine generators,

$$\epsilon_{\pm} = \sum_{i=1}^{n-1} \mu_i E^{(0)}_{\pm\alpha_i} + m_0 E^{(\pm 1)}_{\mp\psi}, \tag{4.1}$$

where ψ is the highest root of $\mathcal{G} = \mathcal{SL}(n)$ and m_0, μ_i with $i = 1 \ldots n-1$ constant parameters, the equations of motion (2.3) reduce to the affine version of the nc Leznov-Saveliev equations (2.15). For this reason we are going to use them again as starting point in order to define the nc extensions of $\widetilde{GL}(n)$ abelian affine Toda theories.

The grading operator Q in the principal gradation for the affine algebra $\widetilde{\mathcal{GL}}(n)$ is taken as

$$Q = \sum_{i=1}^{n-1} \frac{2\lambda_i \cdot H^{(0)}}{\alpha_i^2} + nd, \tag{4.2}$$

where d is the derivation generator and its coefficient is chosen such that this gradation ensures that the zero grade subspace $\mathcal{G}_0$ coincides with its counterpart on the corresponding Lie algebra $\mathcal{SL}(n)$, apart from the generator d. The abelian subalgebra of grade zero is in this case $\mathcal{G}_0 = \{I, h_1^{(0)}, h_2^{(0)}, \ldots h_{n-1}^{(0)}, d\}$ and for this reason the zero grade group element B is parameterized as the $\star$-exponentiation of these generators as in (3.2) [5]. Working in the $n \times n$ representation,

$$(h_i)^{(0)}_{\mu\nu} = \delta_{\mu\nu}(\delta_{i,\mu} - \delta_{i+1,\mu}), \quad (E^{(\pm 1)}_{\alpha_i})_{\mu\nu} = \lambda^{\pm 1}\delta_{\mu,i}\delta_{\nu,i+1}, \quad (E^{(\pm 1)}_{-\alpha_i})_{\mu\nu} = \lambda^{\pm 1}\delta_{\nu,i}\delta_{\mu,i+1},$$

where λ is the spectral parameter and using the variables (3.6), the components of the gauge potentials (2.18) read

$$\begin{aligned} \bar{A}_{ij} &= \bar{\partial}(e_\star^{\phi_i}) \star e_\star^{-\phi_i}\delta_{ij} + \mu_i \delta_{i+1,j} + \lambda m_0 \delta_{i,n}\delta_{j,1}, \\ A_{ij} &= -\mu_i e_\star^{\phi_{i+1}} \star e_\star^{-\phi_i}\delta_{i,j+1} - \frac{m_0}{\lambda} e_\star^{\phi_1} \star e_\star^{-\phi_n}\delta_{i,1}\delta_{j,n}. \end{aligned} \tag{4.3}$$

Introducing the potentials (4.3) in the $\star$-zero-curvature condition (2.17), the n-coupled equations of motion,

$$\partial(\bar{\partial}(e_\star^{\phi_k}) \star e_\star^{-\phi_k}) = \tag{4.4}$$
$$\mu_k^2 e_\star^{\phi_{k+1}} \star e_\star^{-\phi_k} - \mu_{k-1}^2 e_\star^{\phi_k} \star e_\star^{-\phi_{k-1}} + m_0^2(\delta_{n,k} - \delta_{1,k}) e_\star^{\phi_1} \star e_\star^{-\phi_n},$$

[4]See [17] for the ordinary commutative case.

[5]The field associated to the derivation generator d has been set to zero as is usually done for the affine theories.

are found. Note that in the previous expression $\mu_0 = \mu_n = 0$ and $k = 1\ldots n$. Applying the commutative limit, the system of equations,

$$
\begin{aligned}
\bar{\partial}\partial\varphi_i &= \mu_i^2 e^{-k_{ij}\varphi_j} - m_0^2 e^{k_{\psi j}\varphi_j}, \\
n\bar{\partial}\partial\varphi_0 &= 0,
\end{aligned} \tag{4.5}
$$

for the original variables (3.6) is obtained, where $k_{\psi j} = \frac{2\psi\cdot\alpha_j}{\alpha_j^2}$ is the extended Cartan matrix. These are the equations of motion of the affine Toda model plus an additional equation for a free scalar field. The action from where the nc $\widetilde{GL}(n)$ affine equations (4.4) can be derived reads

$$
\begin{aligned}
&S(\phi_1, \ldots, \phi_n) = \\
&\qquad \sum_{k=1}^{n} S_{WZNW_\star}(e_\star^{\phi_k}) + \frac{k}{2\pi}\int d^2z \left(\sum_{k=1}^{n-1} \mu_k^2 e_\star^{\phi_{k+1}} \star e_\star^{-\phi_k} + m_0^2 e_\star^{\phi_1} \star e_\star^{-\phi_n} \right),
\end{aligned} \tag{4.6}
$$

which in the commutative limit reduces to

$$
S(\varphi_1, \ldots, \varphi_{n-1}, \varphi_0) = S_{AT}(\varphi_1, \ldots, \varphi_{n-1}) + nS_0(\varphi_0), \tag{4.7}
$$

with

$$
S_{AT}(\varphi_1, \ldots, \varphi_{n-1}) = -\frac{k}{4\pi}\int d^2z \left(k_{ij}\partial\varphi_i\bar{\partial}\varphi_j - \left(2\sum_{i=1}^{n-1} \mu_i^2 e^{-k_{ij}\varphi_j} + 2m_0^2 e^{k_{\psi_j}\varphi_j} \right)\right), \tag{4.8}
$$

i.e. the usual affine Toda action plus the action for a free field. The complex $\widetilde{SL}(n)$ nc abelian affine Toda field theory (4.6) ($\phi_k \to i\phi_k$) satisfies the usual abelian affine one-soliton solution [26, 40]

$$
i\varphi_k = \ln\left(\frac{1 + e^{\sigma x - \lambda t + \xi + \frac{2\pi i a}{n}k}}{1 + e^{\sigma x - \lambda t + \xi + 2\pi i a}} \right), \tag{4.9}
$$

where we have considered that $\mu_k = m_0 = m$ for $k = 1$ to $n-1$, σ and λ are real parameters satisfying $\sigma^2 - \lambda^2 = 16m^2\sin^2\frac{\pi a}{n}$, a is an integer in the set $\{1, 2, \ldots n-1\}$ and ξ is an arbitrary complex parameter. This is true since we have considered for the extra field φ_0 the solution $\varphi_0 = \sigma x - \lambda t$. In this case due to the particular dependence of ϕ_k on the variables x, t the star product of fields become the usual product. The solitons (4.9) are kinks, whose center of mass is at $\sigma^{-1}(\lambda t - \mathrm{Re}\xi)$, they move with velocity $\frac{\lambda}{\sigma}$ and have the characteristic size σ^{-1}. To the soliton type solutions of (4.6) we can associate a topological charge

$$
Q_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \frac{\partial\varphi_k}{\partial x}, \quad \text{for} \quad k = 1\ldots n-1, \tag{4.10}
$$

which for the specific one-soliton solutions (4.9) will depend on the parameters a and $\mathrm{Im}\xi$. This charge is associated to the discrete symmetry of the model (4.6) $i\phi_k \to i\phi_k + 2i\pi l$ with l an integer number and in fact due to the infinite generalizations possible for the derivative terms there is an ambiguity in its definition. The study of multisolitons solutions of these theories could be possible achieved by a nc version of the dressing method [41]. We expect that at this level the solution for φ_0 would not be trivial. In [8] was outlined the construction of nc multisolitons through this method for the specific case of the sine-Gordon model. Nevertheless the explicit construction of multisoliton solutions for this theory has not been performed yet.

The action (4.6) also has the left-right local symmetry (3.14) and the global $U_L(1) \times U_R(1)$ $e_\star^{\phi_k} \to e^{i\alpha_1} e_\star^{\phi_k} e^{i\alpha_2}$. This point seems interesting since in case of this symmetry being associated to some conserved charge, the soliton type solutions of the nc abelian affine Toda field theories could possible carry electric charge, what it does occur in the usual case.

4.1 NC sinh/sine-Gordon

The sinh/sine-Gordon model, connected with the $\widetilde{\mathcal{SL}}(2)$ loop algebra is the simplest example of an abelian affine Toda theory. In this sense a nc sinh/sine-Gordon extension has been already contemplated in (4.4). Considering $n = 2$, $\phi_1 = \phi_+$, $\phi_2 = \phi_-$ and $m_0 = \mu_1 = \mu$ (4.4) reduced to

$$\begin{aligned} \partial(\bar{\partial}(e_\star^{\phi_+}) \star e_\star^{-\phi_+}) &= \mu^2(e_\star^{\phi_-} \star e_\star^{-\phi_+} - e_\star^{\phi_+} \star e_\star^{-\phi_-}), \\ \partial(\bar{\partial}(e_\star^{\phi_-}) \star e_\star^{-\phi_-}) &= \mu^2(-e_\star^{\phi_-} \star e_\star^{-\phi_+} + e_\star^{\phi_+} \star e_\star^{-\phi_-}). \end{aligned} \tag{4.11}$$

Computing the sum and difference of the previous equations

$$\begin{aligned} &\partial(\bar{\partial}(e_\star^{\phi_+}) \star e_\star^{-\phi_+} + \bar{\partial}(e_\star^{\phi_-}) \star e_\star^{-\phi_-}) = 0, \\ &\partial(\bar{\partial}(e_\star^{\phi_+}) \star e_\star^{-\phi_+} - \bar{\partial}(e_\star^{\phi_-}) \star e_\star^{-\phi_-}) = 2\mu^2(e_\star^{\phi_-} \star e_\star^{-\phi_+} - e_\star^{\phi_+} \star e_\star^{-\phi_-}), \end{aligned} \tag{4.12}$$

an equivalent version of nc sinh-Gordon is obtained. The complex version of this system have been already presented in the literature [8]. There ([8]) the nc extension of sine-Gordon model was constructed through a dimensional reduction process, starting from the linear system of the nc extension of the 2+1 sigma model, which in [44] was shown to be integrable. Here we have seen how this model can be also obtained directly in two-dimensions from the nc extension of the Leznov-Saveliev equations (2.15). The first of these two equations belongs also to the system (3.23) that defines the nc Liouville generalization in this parameterization. In the commutative limit, this equation produces a free field equation for φ_0 and the second one reduces to the usual sinh-Gordon equation

$$\partial\bar{\partial}\varphi_1 + 2\mu^2 \sinh 2\varphi_1 = 0. \tag{4.13}$$

In [42], the version (4.11) of nc sine-Gordon was also derived through a reduction process from the nc affine Toda model coupled to matter fields.

The action (4.6) corresponding to this nc generalization of the sinh-Gordon model reads

$$S(\phi_+,\phi_-) = S_{WZNW_\star}(e_\star^{\phi_+}) + S_{WZNW_\star}(e_\star^{\phi_-}) + \\ + \frac{k}{2\pi}\int d^2z\mu^2(e_\star^{\phi_-}\star e_\star^{-\phi_+} + e_\star^{\phi_+}\star e_\star^{-\phi_-}). \tag{4.14}$$

One can define, in the same way, an equivalent sinh/sine-Gordon model using the alternative parameterization (3.26), i.e.

$$\begin{aligned} \partial(\bar\partial(e_\star^{\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e_\star^{-\varphi_1}) &= \mu^2(e_\star^{-2\varphi_1} - e_\star^{2\varphi_1}), \\ \partial(\bar\partial(e_\star^{-\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e_\star^{\varphi_1}) &= -\mu^2(e_\star^{-2\varphi_1} - e_\star^{2\varphi_1}). \end{aligned} \tag{4.15}$$

Let us now also present the sum and difference of the previous equations

$$\begin{aligned} \partial(\bar\partial(e_\star^{\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e^{-\varphi_1} + \bar\partial(e_\star^{-\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e_\star^{\varphi_1}) &= 0, \\ \partial(\bar\partial(e_\star^{\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e^{-\varphi_1} - \bar\partial(e_\star^{-\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e_\star^{\varphi_1}) &= 2\mu^2(e_\star^{-2\varphi_1} - e_\star^{2\varphi_1}). \end{aligned} \tag{4.16}$$

The complex version of this system ($\varphi_1 \to i\varphi_1, \quad \varphi_0 \to i\varphi_0$) also reproduce another suggestion for nc sine-Gordon presented in [8]. Notice that the first equation of this system belongs to the nc generalization of the Liouville model (3.31) too and it in the commutative limit $\theta \to 0$ becomes a free field equation for φ_0. The second equation of (4.16) in the same limit produces the usual sinh-Gordon equation (4.13).

The action for (4.15) can be obtained from (2.19) with B as in (3.26) and the constant generators $\epsilon_\pm$ as in (4.1), namely

$$S(\varphi_1,\varphi_0) = 2S_{PC_\star}(e_\star^{\varphi_1}) + 2S_{WZNW_\star}(e_\star^{\varphi_0}) + \frac{k}{2\pi}\int d^2z\mu^2(e_\star^{-2\varphi_1} + e_\star^{2\varphi_1}) - \\ -\frac{k}{2\pi}\int d^2z(\bar\partial e_\star^{\varphi_0}\star e_\star^{-\varphi_0}\star(e_\star^{-\varphi_1}\star\partial e_\star^{\varphi_1} + e_\star^{\varphi_1}\star\partial e_\star^{-\varphi_1})). \tag{4.17}$$

As a particular case of (4.9) for $n=2$, the one-soliton

$$i\varphi_1 = 2\tan^{-1}(e^{\sigma x - \lambda t + \xi}) \tag{4.18}$$

is a solution of (4.15) and (4.11) with $\sigma^2 - \lambda^2 = 16m^2$. To compute the scattering amplitudes even for this theory, the simplest of all the affine Toda theories is not so simple. For this reason in [8] the scattering amplitudes were calculated only up to tree level and only for some particle dispersion processes. Apparently no particle production seems to occur what could lead to a factorized S-matrix. In this sense, the nc sine-Gordon model as defined in (4.17) seems to retain the integrability properties of the original model. We expect that the other nc affine Toda theories (those with $n > 2$) will behave in the same way, although the corresponding S-matrices have not been investigated so far.

4.2 About previous suggestions

The nc sine-Gordon by Grisaru-Penati:

$$\begin{aligned}
&\partial(\bar{\partial}(e_\star^{\frac{i}{2}\phi}) \star e_\star^{-\frac{i}{2}\phi} + \bar{\partial}(e_\star^{-\frac{i}{2}\phi}) \star e_\star^{\frac{i}{2}\phi}) = 0, \\
&\partial(\bar{\partial}(e_\star^{\frac{i}{2}\phi}) \star e_\star^{-\frac{i}{2}\phi} - \bar{\partial}(e_\star^{-\frac{i}{2}\phi}) \star e_\star^{\frac{i}{2}\phi}) = 4\mu^2 \sin_\star \phi,
\end{aligned} \tag{4.19}$$

was presented in [5, 6]. The system (4.19) was derived in [6] starting from the nc extension of the Leznov-Saveliev equations (2.15), but excluding of the zero grade subspace the direction of the identity generator. So, essentially that means that we were considering the $\mathcal{SL}(2)$ algebra and the $GL(2)$ group (as the $SL(2)$ group is not closed on the nc setting). From our point of view this fact spoils the integrability properties of the theory. That is why the tree level scattering amplitude of (4.19) suffers from acasual behavior and production of particles occurs [7]. By the other side, we expect that the models deformed, establishing the appropriate *algebra-group* relation, could preserve the integrability properties of their commutative counterparts.

The nc sine-Gordon by Zuevsky:

$$\bar{\partial}\left((e_\star^{\beta\phi})^{-1}_{\star L} \star \partial e_\star^{\beta\phi}\right) = \frac{1}{2}(e_\star^{\beta\phi} - e_\star^{-\beta\phi}), \tag{4.20}$$

presented in [10] is obtained from a generalized zero curvature condition based on continual algebras. In fact this is the corresponding equation of motion of a deformation of a sine-Gordon like action done substituting the product of fields by the star product and generalizing the derivative terms as $\partial\phi \to e_\star^{-\phi} \star \partial e_\star^{\phi}$. So taking $\beta = 1$ it can be derived from the action principle

$$S_{sG-1} = S_{WZNW_\star}(e_\star^{\phi}) + \frac{k}{4\pi}\int d^2z\mu^2(e_\star^{\phi} - e_\star^{-\phi}). \tag{4.21}$$

It happened to be difficult to find for it a zero curvature representation in terms of the generators of the $\widetilde{SL}(2)$ algebra as in the usual case, so it could be interesting to investigate the properties of its S-matrix in order to test its integrability.

4.3 NC sinh/sine-Gordon from nc self-dual Yang-Mills

The nc sinh-Gordon models (4.11) and (4.15) can be derived from nc self-dual Yang-Mills (3.36) considering

$$J = e_\star^{\mu y(E_\alpha + \lambda E_{-\alpha})} \star B \star e_\star^{-\mu\bar{y}(E_{-\alpha} + \frac{1}{\lambda}E_\alpha)}, \tag{4.22}$$

where λ represents the spectral parameter. It is not difficult to see that (3.36) with $x_3 = t$ and $x_4 = ix$, in this case reduces to

$$M \star \{\partial(\bar{\partial}B \star B^{-1}) - \epsilon_+ B\epsilon_- \star B^{-1} + B\epsilon_- \star B^{-1}\epsilon_+\} \star M^{-1} = 0, \tag{4.23}$$

where $M = e_\star^{\mu y(E_\alpha + \lambda E_{-\alpha})}$ and where we have introduced the constant generators $\epsilon_\pm = \mu(E_{\pm\alpha} + E_{\mp\alpha}^{(\pm 1)})$ for $\widetilde{\mathcal{SL}}(2)$. In this way (4.23) represents the nc sine-Gordon extensions (4.11) and (4.15) with the appropriate zero grade element B.

Before we concluded this section let us remark that in the construction of the nc generalizations of abelian and abelian affine Toda theories we have chosen only one type of parameterization for the element B of the zero grade subgroup. Whereas it is possible to use the alternative parameterization

$$B = (\prod_{i=1}^{n-1} e_\star^{\varphi_i h_i^{(0)}}) \star e_\star^{\varphi_0 I}, \tag{4.24}$$

which will lead to generalizations of (3.30) and (4.15) for $n > 2$.

5. NC Toda field theories from nc self-dual Chern-Simons

In [9] was proposed a generalization to the nc plane of the Toda and affine Toda models considering as starting point a nc extension of the Dunne-Jackiw-Pi-Trugenber (DJPT) [29] model of a $U(N)$ Chern-Simons gauge theory coupled to a nonrelativistic complex bosonic matter field on the adjoint representation. The lowest energy solutions of this model satisfy a nc extension of the self-dual Chern-Simons equations from where through a proposed ansatz, the nc generalizations of Toda and affine Toda theories were constructed [9]. Although in the commutative case this procedure will lead to the well known second order differential equations of the Conformal Toda or affine Toda theories [29], in the noncommutative scenario the generalization of Toda theories proposed in [9] were expressed as systems of first order equations which could not be reduced to coupled second order equations in general.

By the other side the self-dual equations for Chern-Simons solitons on nc space can be related to the equation of the $U(N)$ nc chiral model, which apparently can be also solved by a nc extension of the uniton method of Uhlenbeck [32] as stated in [9]. In this way solutions to the proposed nc Toda theories can be explicitly constructed [9]. Particulary in [9] was constructed the simplest solution to the nc generalization of Liouville model as proposed in that work.

In this section we would like to make contact between our nc extensions of the abelian and abelian affine Toda models (3.8), (4.4) and the proposal presented in [9] for these Toda models. We will see in the following that the nc Leznov-Saveliev equations can be also obtained from the nc Chern-Simons self-dual soliton equations. This gives the possibility of obtaining nc extensions of Toda and affine Toda models described by second order differential equations and at the same time it can be possible to use the equivalence of the self-dual equations to the nc chiral model equation to construct solutions of the nc Toda models.

Consider the nc four dimensional (anti-) self-dual Yang-Mills equations for a non-abelian gauge theory in euclidian space

$$F_{12} = -F_{34}, \quad F_{13} = F_{24}, \quad F_{14} = -F_{23}, \tag{5.1}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]_\star$ is the field strength. Considering the covariant derivatives defined as $D_i = \partial_i + [A_i, \;]_\star$ and taking all fields to be independent of x_3 and x_4, the equations (5.1) reduce to

$$F_{12} = -[A_3, A_4]_\star, \quad D_1 A_3 = D_2 A_4, \quad D_1 A_4 = -D_2 A_3, \tag{5.2}$$

As the next step define the covariant derivatives as $D = D_1 + iD_2$, $\bar{D} = D_1 - iD_2$ and the gauge fields as $A = A_1 + iA_2$, $\bar{A} = A_1 - iA_2$ with the coordinates $\tilde{z} = \frac{x_1 - ix_2}{2}$, $\bar{\tilde{z}} = \frac{x_1 + ix_2}{2}$ and the corresponding partial derivatives $\tilde{\partial} = \partial_1 + i\partial_2$, $\bar{\tilde{\partial}} = \partial_1 - i\partial_2$. If now we identify $\Psi = \sqrt{\kappa}(A_3 - iA_4)$ (5.2) transforms to the Chern-Simons self-dual equations,

$$\begin{aligned} \bar{D}\Psi = \bar{\tilde{\partial}}\Psi + [\bar{A}, \Psi]_\star = 0, \\ F_{+-} = \bar{\tilde{\partial}} A - \tilde{\partial}\bar{A} + [\bar{A}, A]_\star = \frac{1}{k}[\Psi^\dagger, \Psi]_\star, \end{aligned} \tag{5.3}$$

where we have considered that $A_i^\dagger = A_i$ for $i = 1 \ldots 4$. The solutions of this system yield static, minimum (zero) energy configurations of a model in $2+1$ dimensions describing charged scalar fields Ψ with nonrelativistic dynamics, minimally coupled to $U(N)$ gauge fields A_μ ($\mu = t, x, y$) with Chern-Simons dynamics in the adjoint representation. The scalar fields Ψ and the gauge fields A_μ take values in the same representation of the gauge Lie algebra. So, the nc self-dual Chern-Simons equations in two dimensions can be obtained through a dimensional reduction from the nc self-dual Yang-Mill equations in four dimensions. In the following we will see how the nc Leznov-Saveliev equations (2.15) can be also obtained from (5.3). For this purpose let us consider that the gauge fields are expressed as

$$A = G^{-1} \star \tilde{\partial} G, \tag{5.4}$$

$$\bar{A} = -A^\dagger, \tag{5.5}$$

where G is an element of the complexification of the gauge group G. Suppose we can decompose G as

$$G = H \star U, \tag{5.6}$$

where H is hermitian and U is unitary. Let us now take our original variables $z, \bar{z}$ which are related to the variables $\tilde{z}, \bar{\tilde{z}}$ through $z = 2\tilde{z}$, $\bar{z} = 2\bar{\tilde{z}}$ and with $x_2 \to ix_2 = -x$ and $x_1 = t$. The field strength is then expressed as

$$F_{+-} = 4U^{-1} \star H \star \bar{\partial}(H^{-2} \star \partial H^2) \star H^{-1} \star U, \tag{5.7}$$

where $H^2 = H \star H$ and $H^{-2} = H^{-1} \star H^{-1}$. The solution of the self-duality equation $\bar{D}\Psi = 0$ is trivially:

$$\Psi = \sqrt{k} G \star \Psi_0(z) \star G^{-1}, \tag{5.8}$$

for any $\Psi_0(z)$. Inserting this solution in the other self-duality equation (5.3) yields the equation for H:

$$4\bar{\partial}(H^{-2} \star \partial H^2) = -\Psi_0 \star H^{-2} \star \Psi_0^\dagger \star H^2 + H^{-2} \star \Psi_0^\dagger \star H^2 \star \Psi_0. \tag{5.9}$$

Consider now that $H^2 = B$, i. e. an element of the zero grade subgroup and $\Psi_0 = 2\epsilon_-$, i.e. the generator of grade -1 which satisfy $\epsilon_-^\dagger = \epsilon_+$. Then the previous equation is written like the nc Leznov-Saveliev equation (2.15). This procedure is a nc extension of an alternative way for deriving the Toda models from the Chern-Simons self-dual equations presented in [29] and which differs from the ansatz used in [9]. Nevertheless both approaches can be mapped one into the another. For instance in [9] was proposed for $U(N)$ the ansatz

$$A = \mathrm{diag}(E_1, E_2, ..., E_N), \quad \tilde{\Psi}_{ij} = \delta_{i,j-1} h_i, \quad i = 1...N-1, \tag{5.10}$$

which after introducing in (5.3) for $\Psi \to \tilde{\Psi}$ leads to a system of coupled first order equations for the $E_1, \ldots, E_N$ fields and for the $h_1, \ldots, h_{N-1}$ fields (see (4.2) and (4.3) of [9]). From (5.4) we see that A is expressed in terms of first order derivatives

$$A = 2U^{-1} \star H^{-1} \star \partial H \star U + 2U^{-1} \star \partial U, \tag{5.11}$$

and

$$\Psi = 2\sqrt{k} U^{-1} \star H \epsilon_- \star H^{-1} \star U. \tag{5.12}$$

With this choice the fields $h_1 \ldots h_{N-1}$ and $E_1 \ldots E_N$ are no longer independent. For $U(N)$ we can also take the constant generators as $\epsilon_\pm = \sum_{i=1}^{n-1} \mu_i E_{\pm\alpha_i}$ and if we consider that $G = G_0$, i.e. the zero grade subgroup it is possible to choose the unitary matrix U in (5.11) and (5.12) as the identity matrix. In this way A in (5.11) will be also a diagonal matrix. More specifically if B is represented by the diagonal matrix

$$B = \mathrm{diag}(g_1, g_2, \ldots g_N), \tag{5.13}$$

with $g_i = e_\star^{\phi_i}$ for $i = 1 \ldots N$, then

$$\Psi_{ij} = \delta_{i,j+1} g_{i+1}^{1/2} \star g_i^{-1/2}, \tag{5.14}$$

where by $g_i^{1/2}$ is understood the function such that $g_i^{1/2} \star g_i^{1/2} = g_i$. In this way we can relate $\Psi = \tilde{\Psi}^\dagger$, then

$$\begin{aligned} \bar{h}_i &= g_{i+1}^{1/2} \star g_i^{-1/2}, \quad \text{for} \quad i = 1 \ldots N-1, \\ E_i &= g_i^{-1/2} \star \partial g_i^{1/2}, \quad \text{for} \quad i = 1 \ldots N. \end{aligned} \tag{5.15}$$

For the affine case the ansatz considered in [9] was

$$A = \mathrm{diag}(E_1, E_2, ...E_N), \tag{5.16}$$
$$\Psi_{ij} = \delta_{i,j-1} h_i, \quad \text{for} \quad i = 1...N-1, \quad \text{except} \quad \text{for} \quad \Psi_{N1} = h_N.$$

Here again we can established relations analogs to (5.15) using (5.11) and (5.12), but now remembering that $\epsilon_\pm = \sum_{i=1}^{N-1} \mu_i E^{(0)}_{\pm\alpha_i} + m_0 E^{(\pm 1)}_{\mp\psi}$. The relations obtained are essentially (5.15), except for the extra component $\Psi_{N1} = h_N = g_N \star g_1^{-1}$ coming from the extra affine generator. In all this process we have combined the constant parameters in such a way that $2\sqrt{k}\mu_i = 1$ for $i = 1 \dots N-1$ and $2\sqrt{k} m_0 = 1$.

We have seen how this way allows to define the nc abelian Toda field theories as second order differential equations from the nc self-dual Chern-Simons equations. Moreover using the relation of the nc Chern-Simons to the nc principal chiral model [9] will be possible to construct the solutions to the nc self-dual equations (5.3) and in this way to the nc Toda models from the solutions of the nc chiral model. In the ordinary commutative case there is a well established procedure to construct the solutions of the chiral model equation with have finite energy called the Uniton method [32]. In [9] was conjectured the extension of this method to the nc plane and was explicitly constructed an specific solution (the simplest) to the nc Liouville model. In a forthcoming work [47] we will present how solutions of our nc extension of Toda models can be obtained by means of this procedure and the relations (5.11) and (5.12). By the other side the possibility of having the nc extensions as second order differential equations with their corresponding actions it is more convenient from the quantization point of view.

6. Conclusions

At the end of [6] was expressed our intention of extending to the nc plane the affine Toda field theories, generalizing in this way the nc sine-Gordon model. With this work we have accomplished this purpose. More specifically we have shown how the nc Leznov-Saveliev equations, proposed in [6], are obtained as the equations of motion of a constrained $WZNW_\star$ model as well as of a constrained two-loop $WZNW_\star$. Starting from these equations, we have extended to the noncommutative plane not only the $\widetilde{\mathcal{GL}}(n)$ abelian affine Toda theories but also the abelian Conformal Toda theories associated to the algebra $\mathcal{GL}(n)$. The actions from which the nc equations of motion of the models can be derived were also presented. As particular examples the nc Liouville and nc sinh/sine-Gordon have been discussed. We have seen how the zero grade subgroup in the nc scenario looses the abelian character. Due to that one can choose alternative parameterization schemes that will lead to equivalent nc extensions of the same model. These two-dimensional theories (nc Liouville and nc sinh/sine-Gordon) can be also obtained from four-dimensional nc

SDYM in the Yang formulation through a suitable dimensional reduction, as we have shown. Furthermore, we have also seen that in general the nc Leznov-Saveliev equations can be obtained from the nc (anti-)SDYM by a dimensional reduction via the nc self-dual Chern-Simons equations. This gives another example in favor of the validity of the Ward conjecture on nc space-time [43].

The construction scheme that we have proposed gives the possibility of extending the models directly in two dimensions without apparently loosing the integrability properties of the original field theory. The crucial point is that the deformation must be done in a consistent way, respecting the *algebra-group* relation, what means that we must extend the group and its corresponding algebra.

We have explicitly studied the relation of our proposal to previous ones establishing in this way the connection with seemingly up to now disconnected versions. Our approach where the nc Leznov-Saveliev equations play a crucial role allows to establish these relations in a relatively simple way and additionally it gives a general framework where many of the previous proposals are included as particular cases. Besides that, our scheme permits the formulation of these theories in form of action principles what is crucial when quantization is intended.

Of course, there are still several interesting directions to pursue in future research. Among them to investigate the full integrability of the nc theories presented which means to construct the conserved charges and to have a more thorough understanding of their influence on the properties of the S-matrix. The symmetries and the multisolitons solutions of the affine models proposed still require a deeper study. Most of the difficulties found in the investigation of these topics are related to the fact that when time is a noncommuting coordinate there is not a nc analog of the Noether theorem. By the other side, since the actions in the nc setup include infinite time derivatives is controversial the definition of the conjugate momenta and in the same way of the corresponding Hamiltonian, Poisson brackets and r-matrices, for example. We hope that the current investigation on this new area of integrability on nc spaces will shed more light on these subjects in the near future.

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7. Appendix A

Let us introduce here some of the algebraic structures used in this paper. The Lie algebra $\mathcal{SL}(n)$ in the Chevalley basis is defined through the commutation relations:

$$
\begin{aligned}
[h_i, h_j] &= \quad 0, \\
[h_i, E_{\alpha_j}] &= \textstyle\sum_{b=1}^{n-1} m_b^{\alpha_j} k_{bi} E_{\alpha_j}, \\
[E_{\alpha_i}, E_{\alpha_j}] &= \begin{cases} \sum_{b=1}^{n-1} l_b^{\alpha_i} h_b, & \text{if} \quad \alpha_i + \alpha_j = 0, \\ \varepsilon(\alpha_i, \alpha_j) E_{\alpha_i + \alpha_j}, & \text{if} \quad \alpha_i + \alpha_j \quad \text{is} \quad \text{a} \quad \text{root}, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned} \tag{7.1}
$$

where $\varepsilon(\alpha_i, \alpha_j)$ are constant such that $\varepsilon(\alpha_i, \alpha_j) = -\varepsilon(\alpha_j, \alpha_i)$, k_{ij} is the Cartan matrix $k_{ij} = \frac{\alpha_i \cdot \alpha_j}{\alpha_j^2}, \quad i = 1 \ldots n-1$, with α_i, α_j in this case simple roots. For any root we have that $\frac{\alpha_i}{\alpha_i^2} = \sum_{b=1}^{n-1} l_b^{\alpha_i} \frac{\alpha_b}{\alpha_b^2} \quad$ and $\quad \alpha_i = \sum_{b=1}^{n-1} m_b^{\alpha_i} \alpha_b$. The bilinear form

$$
\begin{aligned}
Tr(h_i h_j) &= k_{ij}, \\
Tr(E_{\alpha_i} E_{\alpha_j}) &= \frac{2}{|\alpha_i|^2} \delta_{\alpha_i + \alpha_j, 0} \quad \text{for any root}, \\
Tr(E_{\alpha_i} h_j) &= 0,
\end{aligned} \tag{7.2}
$$

is also introduced. The loop algebra $\widetilde{\mathcal{SL}}(n)$ is the Lie algebra of traceless matrices with entries which are Laurent polynomials in λ

$$
\widetilde{\mathcal{SL}}(n) = C(\lambda, \lambda^{-1}) \otimes \mathcal{SL}(n). \tag{7.3}
$$

The structure of the Lie algebra is introduced by the relation

$$
[\lambda^n \otimes T_i, \lambda^m \otimes T_j] = \lambda^{n+m} \otimes f_{ijk} T_k, \tag{7.4}
$$

where $m, n \in \mathbb{Z}$ and the elements of the form $1 \otimes T_i$ are identified with the algebra $\mathcal{SL}(n)$ which is a subalgebra of $\widetilde{\mathcal{SL}}(n)$. In this sense we can write $\lambda^n \otimes T_i$ as $\lambda^n T_i$.

The derivation $d = \lambda \frac{d}{d\lambda}$ generator included in the grading operators acts as

$$
[d, E_{\alpha_i}^{(n)}] = n E_{\alpha_i}^{(n)}, \quad [d, h_i^{(n)}] = n h_i^{(n)}. \tag{7.5}
$$

We have also considered the symmetric bilinear form

$$
Tr(h_i^{(m)} h_j^{(n)}) = k_{ij} \delta_{m+n,0}, \tag{7.6}
$$

$$
Tr(E_{\alpha_i}^{(m)} E_{\alpha_j}^{(n)}) = \frac{2}{|\alpha_i^2|} \delta_{\alpha_i + \alpha_j, 0} \delta_{m+n,0}, \tag{7.7}
$$

and that $|\alpha_i|^2 = 2$ for simple roots.

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